MA’RUZA

ANIQMAS INTEGRALNI HISOBLASH USULLARI. KVADRAT UCHHAD QATNASHGAN FUNKSIYALARNI INTEGRALLASH.

Mavzuning rejasi

1. Bevosita integrallash.
2. O’zgaruvchilarni almashtirish yordamida integrallash.
3. Bo’laklab integrallash.
4. Kvadrat uchhad qatnashgan funksiyalarni integrallash.

Tayanch so’z va iboralar: bevosita integrallash, o’zgaruvchilarni almashtirish, jadval integrali, boshlang’ich funksiya, funksiya differensiali, yangi o’zgaruvchi, bo’laklab integrallash, ko’paytmani differensiallash, kvadrat uchhad qatnashgan integral, to’la kvadratini ajratish.

1. Bevosita integrallash.

Aniqmas integralni hisoblashda integral ostidagi funksiyaning boshling’ich funksiyasi topiladi. Bu boshlang’ich funksiya yuqorida keltirilgan integral xossalaridan hamda integrallar jadvalidan foydalanib topiladi, bunga bevosita integrallash deyiladi. Bundan tashqari integrallashda o’zgaruvchini almashtirish va bo’laklab integrallash usullaridan foydalaniladi.

2. O’zgaruvchini almashtirish yoki o’rniga qo’yish usuli.

Bu usul bilan integrallashda o’zgaruvchi x yangi o’zgaruvchi t bilan ma’lum munosabatda shunday almashtiriladiki, natijada oddiy integralga ega bo’lamiz.

Bizga $\int f(x)dx$ berilgan bo’lsin, $x=\varphi(t)$ almashtirishni olaylik. Bundan $dx=\varphi'(t)dt$ ni topib, uni berilgan integralga qo’ysak. Qo’yidagi ifoda hosil bo’ladi.

$$\int f(x)dx=\int f(\varphi(t))\varphi'(t)dt$$

Bu esa berilgan integralga nisbatan ancha sodda bo’ladi. Umuman, integralni hisoblaganda turli almashtirishlar yordami bilan berilgan integral, jadvaldagi integrallardan birortasiga keltiriladi. So’ngra, jadvaldan boshlang’ich funksiya topiladi.

Ba’zan, berilgan integral $\int \frac{\varphi'(x)}{\varphi(x)}dx$ ko’rinishda berilgan bo’lsa, bunda $t=\varphi(x)$ almashtirish bilan integral juda soddalashadi. Haqiqatan, $t=\varphi(x)$, $dt=\varphi'(x)dx$

$$\int \frac{\varphi'(x)}{\varphi(x)}dx=\int \frac{dt}{t}=\ln|t|+C=\ln|\varphi(x)|+C$$

Bundan ko’rinadiki o’zgaruvchini almashtirish bilan integrallanganda chiqqan natija yana avvalgi o’zgaruvchi yordamida ifodalanar ekan, ya’ni t o’zgaruvchidan x o’zgaruvchiga o’tilar ekan.

Misol. Qo’yidagi integral hisoblansin: $\int \frac{\sin x dx}{\sqrt{1+2\cos x}}$, bunda $1+2\cos x=t$ deb olamiz. Bu holda $-2\sin x dx=dt$ bo’ladi. Demak,

$$\int \frac{\sin x dx}{\sqrt{1+2\cos x}}=-\frac{1}{2}\int \frac{dt}{\sqrt{t}}=-\frac{1}{2}\int t^{-\frac{1}{2}}dt=-\frac{1}{2}\cdot 2t^{\frac{1}{2}}+C=$$

$$=-\sqrt{t}+C=-\sqrt{1+2\cos x}+C.$$

3. Bo'laklab integrallash usuli.

Bizga ikkita differensiallanuvchi $u(x)$ va $v(x)$ funksiyalar berilgan bo'lsin. Bu funksiyalar ko'paytmasi $(u \cdot v)$ ning differensialini topamiz. Ma'lumki, ko'paytmaning differensiali formulasidan $d(uv) = du \cdot v + u \cdot dv$

Bu ifodani har ikkala tomonini integrallab, qo'yidagini topamiz.

$uv = \int vdu + \int udv$ yoki $\int udv = uv - \int vdu$ (1)

Oxirgi topilgan formula bo'laklab integrallash formulasi deyiladi. Bu formulani qo'llab integral hisoblanganda $\int udv$ ko'rinishdagi integral, ancha soda bo'lgan $\int vdu$ ko'rinishdagi integralga keltiriladi. Agar integral ostida $y = \ln x$ funksiya, yoki ikkita funksiyaning ko'paytmasi hamda teskari trigonometrik funksiyalar qatnashgan bo'lsa, bunda bo'laklab integrallash formulasi qo'llaniladi. Bu usul bilan integrallaganda yangi o'zgaruvchiga o'tishga hojat bo'lmaydi.

Umuman, aniqmas integralni hisoblashda topilgan natija yoniga o'zgarmas $C = const$ ni qo'shib qo'yish shart. Aks holda integralni bitta qiymati topilib, qolganlarini tashlab yubogan bo'lamiz. Bu esa integrallashda xatolikka yo'l qo'yilgan deb hisoblanadi.

Misol. $\int xarctgxdx$ ni hisoblang.

$$u = arctgx, \quad dv = xdx, \quad du = \frac{dx}{1+x^2}, \quad v = \int xdx = \frac{x^2}{2}$$

(bunda $C = 0$ deb olamiz) (1) formulani qo'llab

$$\int x \cdot arctgxdx = \frac{x^2}{2} arctgx - \int \frac{x^2}{2(1+x^2)} dx =$$

$$= \frac{x^2}{2} arctgx - \frac{1}{2} x + \frac{1}{2} arctgx + C = \frac{x^2+1}{2} arctgx - \frac{x}{2} + C .$$

3. Kvadrat uchhad qatnashgan funksiyalarni integrallash.

Bunday integrallarga asosan quyidagi integrallar kiradi.

1. $\int \frac{dx}{ax^2+bx+c}$. 2. $\int \frac{Ax+B}{ax^2+bx+c} dx$. 3. $\int \frac{dx}{\sqrt{ax^2+bx+c}}$.

4. $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}} dx$. 5. $\int \sqrt{ax^2+bx+c}dx$ va 6. $\int (ax^2+bx+c)dx$.

Bunday integrallarni hisoblash uchun integral ostida qatnashgan uchhadni to'la kvadratga ajratib, ikki had kvadratining algebraik yig'indisiga keltiriladi. Natijada hosil bo'lgan ifodani integrallar jadvali yordamida integrallash mumkin bo'ladi. Kvadrat uchhadning to'liq kvadrati quyidagicha ajratiladi:

$$ax^2+bx+c = a(x^2 + \frac{b}{a}x + \frac{c}{a}) = a\left[(x+\frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right] =$$

$$= a\left[(x+\frac{b}{2a})^2 \pm k^2 \right] \text{ (bu yerda } \pm k^2 = \frac{b^2-4ac}{4a^2} \text{)}$$

Bu yerdagi plyus yoki minus ishora ax^2+bx+c kvadrat uch hadning ildizlari haqiqiy yoki kompleks bo'lishiga qarab aniqlanadi, ya'ni b^2-4ac ni ishorasiga qarab. To'liq kvadrati ajratilgandan keyin yuqorida keltirilgan integrallarni mos ravishda I_1, I_2, I_3, I_4, I_5 va I_6 lar bilan belgilasak, quyidagi ko'rinishga ega bo'ladi.

1. $I_1 = \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{(x+\frac{b}{2a})^2 \pm k^2}$. Bunda $x + \frac{b}{2a} = t,\ dx = dt$ desak,

$I_1 = \int \frac{dt}{t^2 \pm k^2}$ ko'rinishga keladi, bu esa jadvaldagi integral.

1-Misol. a) $\int \frac{dx}{2x^2+8x+20}$ hisoblansin.

Yechish. $\int \frac{dx}{2x^2+8x+20} = \frac{1}{2}\int \frac{dx}{x^2+4x+10} = \frac{1}{2}\int \frac{dx}{x^2+4x+4+10-4} =$

$= \frac{1}{2}\int \frac{dx}{(x+2)^2} = I_1$. $x+2=t,\ dx=dt$ bo'ganidan.

$I_1 = \frac{1}{2}\int \frac{dt}{t^2+2} = \frac{1}{2}\frac{1}{\sqrt{6}} arctg \frac{t}{\sqrt{6}} + C$. t o'rniga x orqali eski ifodani qo'yib, oxirgi natijani topamiz: $I_1 = \frac{1}{2\sqrt{6}} arctg \frac{x+2}{\sqrt{6}} + C$.

2. $I_2 = \int \frac{Ax+B}{ax^2+bx+c}dx = \int \frac{\frac{A}{2a}(2ax+b)(B-\frac{Ab}{2a})}{ax^2+ba+c}dx =$

$= \frac{A}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \left(B-\frac{Ab}{2a}\right)\int \frac{dx}{ax^2+bx+c}$ ikkita integralga keltirib hisoblanadi, ularni I_1^* va I_1^{**} bilan belgilab, qo'yidagicha hisoblaymiz.

$I_1^* = \int \frac{(2ax+b)}{ax^2+bx+c}dx = \begin{bmatrix} ax^2+bx+c=t \\ (2ax+b)dx=dt \end{bmatrix} = \int \frac{dt}{t} = \ln|t| + C =$

$= \ln\left|ax^2+bx+c\right| + C$ (bu yerda c kichik, C esa katta).

$I_1^{**} = \left(B-\frac{Ab}{2a}\right)\int \frac{dx}{ax^2+bx+c} = \left(B-\frac{Ab}{2a}\right)I_1$ edi. Shuning uchun I_2 ko'rinishdagi integral quyidagicha hisoblanar ekan $I_2 = I_1^* + I_1^{**} = \frac{A}{2a}\ln\left|ax^2+bx+c\right| + \left(B-\frac{Ab}{2a}\right)I_1$

2-Misol. $I_2 = \int \frac{x+3}{x^2-2x-5}dx$ integralni hisoblang.

Yechish. $I_2 = \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+(3+\frac{2\cdot 1}{2})}{x^2-2x-5}dx =$

$\frac{5}{2}\int \frac{(2x+2)}{x^2-2x-5}dx + 4\int \frac{dx}{x^2-2x-5} = \frac{1}{2}\ln\left|x^2-2x-5\right| + 4\int \frac{dx}{(x-1)^2-6} =$

$= \frac{1}{2}\ln\left|x^2-2x-5\right| + 2\cdot\frac{1}{\sqrt{6}}\ln\left|\frac{\sqrt{6}-(x-1)}{\sqrt{6}+(x-1)}\right| + C$.

Demak, $I_2 = \frac{1}{2}\ln\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\ln\left|\frac{\sqrt{6}-(x-1)}{\sqrt{6}+(x-1)}\right| + C$ bo'ladi.

3. $I_3 = \int \frac{dx}{\sqrt{ax^2+bx+c}}$. Bu integralni yuqoridagi qo'llanilgan almashtirishlar yordamida quyidagi ko'rinishga keltiramiz:

$a>0$ bo'lganda $I_3 = \int \frac{dx}{\sqrt{t^2 \pm k^2}}$. $a<0$ bo'lganda $I_3 = \int \frac{dx}{\sqrt{k^2-t^2}}$, bular esa jadvaldagi integrallardan iborat.

3-Misol. $I_3 = \int \frac{dx}{\sqrt{x^2-4x-3}}$ integral hisoblansin. $x^2-4x-3=(x-2)^2-7$

$dx = d(x-2)$, u holda $I_3 = \int \frac{dx}{\sqrt{x^2-4x-3}} = \int \frac{d(x-2)}{\sqrt{(x-2)^2-7}} = \ln\left|x-2+\sqrt{(x-2)^2-7}\right| + C$.

Jadvaldagi integralga asosan hisobladik.

4. $I_4 = \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx = \int \frac{\frac{A}{2a}(2ax+b)+(B-\frac{Ab}{2a})}{\sqrt{ax^2+bx+c}}dx =$

$= \frac{A}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + \left(B-\frac{Ab}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$ ikkita integralga ajratib, ularni I_4^* va I_4^{**} bilan belgilab quyidagicha hisoblaymiz:

$I_4^* = \frac{A}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx = \begin{bmatrix} ax^2+bx+c=t \\ (2ax+b)dx=dt \end{bmatrix} = \frac{A}{2a}\int \frac{dt}{\sqrt{t}} = \frac{A}{2a}\int \frac{dt}{\sqrt{t}} = \frac{A}{2a}a\sqrt{t}+C =$

$= \frac{A}{a}\sqrt{ax^2+bx+c}+C$.

$I_4^{**} = \left(B-\frac{Ab}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}} = \left(B-\frac{Ab}{2a}\right)I_3$ bo'lganligidan

$I_4 = I_4^* + I_4^{**} = \frac{A}{a}\sqrt{ax^2+ba+c} + \left(B-\frac{Ab}{2a}\right)I_3$ bo'ladi.

4-Misol. $I_4 = \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$ integralni hisoblang.

Yechish. $I_4 = \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)+(3-10)}{\sqrt{x^2+4x+10}}dx =$

$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{dx}{\sqrt{(x+2)^2+6}} = 5\sqrt{x^2+4x+10} -$

$-7\ln\left|x+2+\sqrt{(x+2)^2+6}\right| + C = 5\sqrt{x^2+4x+10} - 7\ln\left|x+2+\sqrt{x^2+4x+10}\right| + C$.

5. $I_5 = \int \sqrt{ax^2+bx+c}dx = \int \sqrt{a(x+\frac{b}{2a})^2 \pm k^2}dx =$

$= \left[\frac{b^2-4ac}{4a^2} = \pm k^2,\ x+\frac{b}{2a} = t,\ dx = dt\right]$ (deb olsak) $= \int \sqrt{a(t^2 \pm k^2}dt$ bu integral esa quyidagi formulalar yordamida hisoblanadi:

I. $\int \sqrt{t^2+b}dt = \frac{1}{2}\sqrt{t^2+b} + \frac{b}{2}\ln\left|t+\sqrt{t^2+b}\right| + C$.

II. $\int \sqrt{a^2-t^2}dt = \frac{1}{2}\sqrt{a^2-t^2} + \frac{a^2}{2}\arcsin\frac{t}{a} + C$.

Misol 5. $\int \sqrt{x^2+2x+6}dx$ integralni hisoblang .

Yechish. Hisoblashda $x^2+2x+6=(x+1)^2+5$ to'la kvadratini ajratib $t=(x+1)$ almashtirish olib, $d(x+1)=dt,\ b=5$ belgilashdan keyin I formula yordamida topamiz

$\int \sqrt{x^2+2x+6}dx = \int \sqrt{(x+1)^2+5}d(x+1) = \frac{x+1}{2}\sqrt{(x+1)^2+5} +$

$+ \frac{5}{2}\ln\left|x+1+\sqrt{(x+1)^2+5}\right| + C$.

6. $I_6 = \int (ax^2+bx+c)dx = a\int \left[(x+\frac{b}{2a})^2 \pm k^2\right]dx =$

$= \left[x+\frac{b}{2a} = t,\ dx = dt\right] = a\int (t^2 \pm k^2)dt$ jadval integraliga keltirib hisoblanadi.